

On the existence of maximizers for functionals with critical exponential growth in \mathbb{R}^2

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Abstract We investigate the problem of existence of a maximizer for

$$S(\alpha, 4\pi) = \sup_{\|u\|=1} \int_B (e^{4\pi u^2} - 1) |x|^\alpha dx,$$

where B is the unit disk in \mathbb{R}^2 and $\alpha > 0$. We prove that supremum is attained for α small.

Key words: Extremal functions, symmetrization, Hénon type equation, critical growth.

1 Introduction

Let $H_0^1(\Omega)$ be the Sobolev space over a bounded domain $\Omega \subset \mathbb{R}^N$, with Dirichlet norm $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$. The Sobolev embedding theorem states that $L^p(\Omega) \subset H_0^1(\Omega)$, for $1 \leq p \leq 2^* = \frac{2N}{N-2}$; equivalently, if we set

$$S_N(p) = \sup_{\|u\| \leq 1} \int_\Omega |u|^p dx,$$

then

$$\begin{aligned} S_N(p) &< \infty, & \text{for } 1 < p \leq 2^* = \frac{2N}{N-2}; \\ S_N(p) &= \infty, & \text{for } p > 2^*; \end{aligned}$$

furthermore, the value of the best Sobolev constant $S_N(2^*)$ is explicit, independent of the domain Ω and it is known that it is never attained in any bounded smooth domain. The maximal growth $|u|^{2^*}$ allowed is called “critical” Sobolev growth. If $N = 2$, every polynomial growth is admitted, but it is easy to show that $H_0^1(\Omega) \not\subset L^\infty(\Omega)$: in this case it is well known that the maximal growth allowed to a function $g : \mathbb{R} \rightarrow \mathbb{R}^+$ such that $\sup_{\|u\| \leq 1} \int g(u) < \infty$ is of exponential type. More precisely, the Trudinger Moser inequality states that for bounded

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domain $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} \sup_{\|u\| \leq 1} \int_{\Omega} e^{\gamma u^2} dx &\leq C(\gamma)|\Omega| \leq C(4\pi)|\Omega|, \text{ for } \gamma \leq 4\pi; \\ \sup_{\|u\| \leq 1} \int_{\Omega} e^{\gamma u^2} dx &= \infty, \text{ for } \gamma > 4\pi, \end{aligned}$$

see [Po], [Tr] and [M]. In contrast with the Sobolev case, the value $C(4\pi)$ is attained when $\Omega = B_1(0)$ is the unit ball in \mathbb{R}^2 , as proved in an interesting paper by Carleson and Chang [CC] (see also [dFdOR]). This result was extended to general bounded domains in \mathbb{R}^2 by Flucher [F].

In this paper we consider the maximization problem

$$S(\alpha, 4\pi) = \sup_{\|u\|=1} \int_B \left(e^{4\pi u^2} - 1 \right) |x|^\alpha dx, \quad (1)$$

where $\alpha > 0$ and B is the unit ball in \mathbb{R}^2 . Here we give a partial answer to a question proposed by Secchi and Serra in a recent paper (see [SS]): is the supremum attained for any $\alpha > 0$? Our main result states that $S(\alpha, 4\pi)$ is attained, at least if the parameter α is small enough.

Problem (1) can be seen as a natural two-dimensional extension of the Hénon-type problem

$$\sup_{\|u\|=1} \left(\int_B |u|^p |x|^\alpha dx \right)^{2/p} = \sup_{u \neq 0} \frac{\left(\int_B |u|^p |x|^\alpha dx \right)^{2/p}}{\int_B |\nabla u|^2 dx} \quad (2)$$

in \mathbb{R}^N with $N \geq 3$ and $1 < p < 2^*$, which has been widely investigated in the last few years. It is easy to verify that (2) is achieved at least by a positive function; since the quotient in (2) is invariant under rotations, it is natural to ask if the supremum is achieved by a radial function. A very interesting result obtained by Smets, Su and Willem ([SSW]) shows that a symmetry breaking phenomenon occurs for any $p \in (2, 2^*)$: in details, for every p in the subcritical range the supremum in (2) is attained by a non radial function when $\alpha \rightarrow \infty$. This result has generated a line of research on the Hénon-type equations (see references in [SS]). On the contrary, the Hénon-type problem in \mathbb{R}^2 with exponential nonlinearities seems to have been much less studied. Very recently, Calanchi and Terraneo (see [CT]) proved some results about the existence of non radial maximizers for the variational problem

$$\sup_{\|u\|=1} \int_B \left(e^{\gamma u^2} - 1 - \gamma u^2 \right) |x|^\alpha dx$$

where $\alpha > 0$ and $0 < \gamma < 4\pi$; in the same line is the work by Secchi and Serra, [SS], where the authors prove a symmetry breaking result for problem

$$\sup_{\|u\|=1} \int_B \left(e^{\gamma u^2} - 1 \right) |x|^\alpha dx.$$

In both papers the authors consider only subcritical exponential growth: in this case, indeed, the existence of a maximizer can be proved with standard arguments (see [SS], proof of Proposition 1). On the contrary, in the critical case, that is, when $\gamma = 4\pi$, it is not clear if the supremum $S(\alpha, 4\pi)$ is attained or not. On one hand, it seems not possible to adapt the proof suggested by Secchi and Serra, which deeply depends on the hypothesis of subcritical growth. On the other hand, due to the presence of the weight $|x|^\alpha$ in front of the nonlinearity, problem (1) cannot be reduced to a one dimensional problem using the technique of Schwarz symmetrization, as proposed by Carleson and Chang.

Our result depends on a new notion of symmetrization, the so called *spherical symmetrization with respect to a measure*, which is the counterpart of Schwarz symmetrization in the unweighted problem. Although we symmetrize with respect to a measure μ which is different from the Lebesgue one, a result by Schulz and Vera de Serio (see [SVS]) states that the gradient norm does not increase, as in the classical case (the result is valid only in \mathbb{R}^2 and with suitable assumption on the measure μ). This fact allows us to adapt the proof presented by de Figueiredo, do Ó and Ruf in [dFdOR], obtaining the following result.

Theorem 1.1. *There exists $\alpha_* > 0$ such that for every $\alpha \in (0, \alpha_*)$, $S(\alpha, 4\pi)$ is attained.*

We remark that the notion of symmetrization with respect to the measure $\int_B |x|^\alpha dx$ gives also a geometric interpretation of the changes of variable performed by Smets, Su and Willem, and later by Secchi and Serra, when dealing with radial functions: see Remark 1 at the end of Section 3.

2 Symmetrization with respect to a measure

In this section we recall the main definitions and properties of symmetrization: we refer to [K] or to [Ba]. We start by a review of the standard definitions. Let Ω be a bounded domain in \mathbb{R}^2 . We denote by $|\Omega|$ the Lebesgue measure of Ω and by $\mathcal{L}_0(\Omega)$ the set of Lebesgue measurable functions defined in Ω up to a.e. equivalence. For every function $u \in \mathcal{L}_0(\Omega)$, we define the distribution function ϕ_u of u by the formula

$$\phi_u(t) = |\{x \in \Omega : |u(x)| > t\}|.$$

A measurable function u in \mathbb{R}^n is called *radially symmetric*, or radial, for short, if $u(x) = \tilde{u}(r)$, $r = |x|$; it is called *rearranged* if it is nonnegative, radially symmetric and \tilde{u} is a non-increasing function of $r > 0$; we also impose that $\tilde{u}(r)$ be right-continuous. We will write $u(x) = u(r)$ by abuse of notation. The *spherical symmetric rearrangement* u^* of u is the unique rearranged function defined in Ω^* which has the same distribution function as u , that is, for every $t > 0$

$$\phi_u(t) = |\{x \in \Omega : |u(x)| > t\}| = \phi_{u^*}(t) = |\{x \in \Omega^* : |u^*(x)| > t\}|,$$

where $\Omega^* = B_R(0)$ is the ball having the same volume as Ω , i.e. $|\Omega| = \omega_n R^n$ (here ω_n is the volume of the unit sphere in \mathbb{R}^n). Then,

$$\begin{aligned} u^*(x) &= \inf \{t > 0 : \phi_u(t) \leq \omega_n |x|^n\} \\ &= \sup \{t > 0 : \phi_u(t) > \omega_n |x|^n\}. \end{aligned} \quad (3)$$

A rearranged function coincides with its spherical rearrangement. Since the distribution functions of u and u^* are identical, $\int_{\Omega} |u|^p dx = \int_{\Omega^*} (u^*)^p dx$ for every $p \in [1, +\infty)$; moreover, for every nonnegative, increasing and left-continuous real function Φ

$$\int_{\Omega} \Phi(u) dx = \int_{\Omega^*} \Phi(u^*) dx.$$

Finally, if $u \in W_0^{1,p}(\Omega)$, then $u^* \in W_0^{1,p}(\Omega^*)$ and

$$\int_{\Omega^*} |\nabla u^*|^p dx \leq \int_{\Omega} |\nabla u|^p dx, \quad (4)$$

for $p \in (0, +\infty)$: this is the celebrated Polya-Szegö inequality.

As a natural generalization of the spherical symmetrization (or Schwarz symmetrization), one can introduce the *spherical symmetrization with respect to a measure* μ defined on the domain Ω . We refer to [SVS]. Let $p : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a nonnegative, measurable and locally integrable function, and consider the absolutely continuous measure μ given by

$$\mu(A) = \int_A p dx$$

for any Lebesgue measurable set A in \mathbb{R}^n . The distribution function $\phi_{\mu,u}$ of u with respect to the measure μ is given by

$$\phi_{\mu,u}(t) = \mu(\{x \in \Omega : |u(x)| > t\});$$

as in the classical case, $\phi_{\mu,u}$ is a monotone, non-increasing and right continuous function. The *spherical symmetric rearrangement* u_{μ}^* of u with respect to the measure μ is the unique rearranged function defined in Ω_{μ}^* whose (classical) distribution function is the same as the distribution function (with respect to the measure μ) of u ; that is, for every $t > 0$

$$\phi_{\mu,u}(t) = \mu(\{x \in \Omega : |u(x)| > t\}) = \phi_{u^*}(t) = |\{x \in \Omega_{\mu}^* : |u^*(x)| > t\}|,$$

where $\Omega_{\mu}^* = B_R(0)$ is the ball centered at the origin with $\mu(\Omega) = |\Omega_{\mu}^*| = \omega_n R^n$. Then,

$$\begin{aligned} u_{\mu}^*(x) &= \inf \{t > 0 : \phi_{\mu,u}(t) \leq \omega_n |x|^n\} \\ &= \sup \{t > 0 : \phi_{\mu,u}(t) > \omega_n |x|^n\}. \end{aligned} \quad (5)$$

Obviously, the spherical symmetric rearrangement $u_{\mathcal{L}}^*$ with respect to the Lebesgue measure is the classical symmetric rearrangement by Schwarz. However, if

μ is not the Lebesgue measure, a rearranged function, in the sense defined above, will not coincide with its μ -rearrangement u_μ^* , since an extra contraction/dilation will take place. In particular, if we consider the density function $p_\alpha(x) = |x|^\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^+$ with $\alpha > 0$ and the associated measure

$$\mu_\alpha(A) = \int_A |x|^\alpha dx \quad (6)$$

defined on the unit sphere B in \mathbb{R}^n , then the μ_α -rearrangement of a rearranged function $u(r)$ (that is, of a non-negative, radial and non-increasing function u) is defined by the formula

$$u_\alpha^*(r) = u \left(r^{\frac{n}{\alpha+n}} \sqrt[n]{\frac{\alpha+n}{n}} \right), \quad (7)$$

where $r \in B \left(0, \sqrt[n]{\frac{\alpha+n}{n}} \right)$.

As in the classical case, for every nonnegative, increasing and left-continuous real function Φ

$$\int_\Omega \Phi(u) d\mu = \int_{\Omega_\mu^*} \Phi(u_\mu^*) dx,$$

so that $\|u\|_{L^p(\Omega, \mu)} = \|u_\mu^*\|_{L^p(\Omega_\mu^*, \mathcal{L})}$ for every $p \in [1, +\infty)$. As regard the gradient norm, it is not known (to our knowledge) if the Polya-Szegő inequality can be maintained for all μ -rearrangements and for $p > 0, n \geq 1$. With the assumptions stated above on μ , if $u \in W^{1,1}(\mathbb{R}^n)$, then $u_\mu^* \in W^{1,1}(\mathbb{R}^n)$; furthermore, Schulz and Vera de Serio have proved the following result:

Theorem 2.1 (F. Schulz, V. Vera de Serio). *Let $p \in \mathcal{C}^0(\bar{D})$ be a nonnegative function on a simply-connected domain D such that $\log p$ is subharmonic where $p > 0$; suppose that $u \in W^{1,2}(\mathbb{R}^2)$ is a non-negative function with compact support in D . Then $u_\mu^* \in W^{1,2}(\mathbb{R}^2)$, and the inequality*

$$\int_{\mathbb{R}^2} |\nabla u_\mu^*|^2 dx \leq \int_{\mathbb{R}^2} |\nabla u|^2 dx \quad (8)$$

holds.

We remark that Theorem 2.1 states that the gradient of the μ -rearrangement does not increase in the $L^2(\mathbb{R}^2)$ norm (that is, considering \mathbb{R}^2 endowed with the Lebesgue measure); different results can be found in [Ta] and in [SW] where a similar inequality is obtained for the $L^2(\mathbb{R}^2, \mu)$ norms.

3 Existence of a maximizer for $S(\alpha, 4\pi)$

This section is devoted to the proof of our main result, Theorem 1.1. As well known, in the “unweighted” case $\alpha = 0$ the supremum $S(0, 4\pi)$ is attained: this is the celebrated result due to Carleson and Chang [CC]. In the subcritical case

$\gamma < 4\pi$, $S(\alpha, \gamma)$ is still attained, as pointed out by Serra and Secchi in [SS] (and the proof is quite easy), whereas in the supercritical case $\gamma > 4\pi$, $S(\alpha, \gamma) = +\infty$ for every $\alpha > 0$, as proved by Calanchi and Terraneo [CT] testing with a suitable sequence of (radial) functions.

The critical case $\gamma = 4\pi$ is more delicate. If we consider the radial version of the maximization problem (1), that is,

$$S^{\text{rad}}(\alpha, \gamma) = \sup_{\substack{u \in H_{0,\text{rad}}^1(B) \\ \|u\| = 1}} \int_B \left(e^{\gamma u^2} - 1 \right) |x|^\alpha dx, \quad (9)$$

it is not hard to prove that the problem is still “subcritical”, provided that $\gamma < 4\pi + 2\pi\alpha$, as proved by Secchi and Serra in [SS]. More in details, they proved that

$$S^{\text{rad}}(\alpha, 4\pi) = \frac{2}{\alpha + 2} S\left(0, 4\pi \frac{2}{\alpha + 2}\right)$$

and standard arguments show that $S(0, 4\pi \frac{2}{\alpha+2})$ is actually attained by a radial function. See also the remark at the end of Section 3 in [CT].

On the contrary, it seems not possible to reduce the problem of maximization of $S(\alpha, 4\pi)$ in the general case, that is, considering also non radial functions, to a subcritical one. Our proof follows the same idea of the one given by de Figueiredo, do Ó and Ruf in [dFdOR] (which differs from the original proof of Carleson and Chang by the use of the concentration-compactness principle). Here is a short outline of the proof:

- if $S(\alpha, 4\pi)$ is not attained, then by the concentration-compactness alternative of P.L. Lions there is a normalized maximizing and concentrating sequence v_n ;
- by means of symmetrization with respect to the measure $\mu_\alpha = \int |x|^\alpha dx$, one can prove an upper bound for any normalized concentrating sequences u_n :

$$\overline{\lim}_{n \rightarrow +\infty} \int_{B_\alpha^*} \left(e^{4\pi |u_n|^2} - 1 \right) dx \leq \frac{2}{\alpha + 2} \pi e$$

- give an explicit function $\omega \in H_0^1(B)$ such that $\|\omega\| = 1$ and

$$\int_B \left(e^{4\pi \omega^2} - 1 \right) |x|^\alpha dx > \frac{2}{\alpha + 2} \pi e.$$

It is clear, then, that the notion of spherical symmetrization with respect to a measure is the fundamental tool which allows to reduce the weighted problem $S(\alpha, 4\pi)$ to a one dimensional problem. See also the remarks at the end of the proof.

First of all, let us recall the concentration-compactness result by P.L. Lions [L] (adapted to the 2-dimensional case):

Proposition 3.1 (P.L. Lions). *Let Ω be a bounded domain in \mathbb{R}^2 , and let $\{u_n\}$ be a sequence in $H_0^1(\Omega)$ such that $\|u_n\|_{H_0^1} \leq 1$ for all n . We may suppose that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, $|\nabla u_n|^2 \rightarrow \nu$ weakly in measure. Then either*
(i) $\nu = \delta_{x_0}$, the Dirac measure of mass 1 concentrated at some $x_0 \in \bar{\Omega}$, and $u \equiv 0$,
or
(ii) there exists $\beta > 4\pi$ such that the family $v_n = e^{u_n^2}$ is uniformly bounded in $L^\beta(\Omega)$ and thus $\int_\Omega e^{4\pi u_n^2} \rightarrow \int_\Omega e^{4\pi u^2}$ as $n \rightarrow +\infty$. In particular, this is the case if u is different from 0.

Proof of Theorem 1.1

Proof. We follow [dFdOR]. We say that a sequence $\{u_n\} \subset H_0^1(B)$ is a *normalized concentrating sequence* if

- i) $\|u_n\|_{H_0^1} = 1$
- ii) $u_n \rightharpoonup 0$ weakly in $H_0^1(B)$
- iii) $\exists x_0 \in B$ such that $\forall \rho > 0$, $\int_{B \setminus B_\rho(x_0)} |\nabla u_n|^2 dx \rightarrow 0$.

Let us suppose that $\{u_n\}$, $\|u_n\| = 1$, is a maximizing sequence for (1), that is, $\lim_{n \rightarrow +\infty} \int_B (e^{4\pi u_n^2} - 1) |x|^\alpha dx = S(\alpha, 4\pi)$. Then, by the concentration-compactness alternative of P.L. Lions, either $\{u_n\}$ is a normalized concentrating (and maximizing) sequence, or $S(\alpha, 4\pi)$ is attained. To conclude the proof, we proceed by the following steps:

- 1) if $\{u_n\}$ is any normalized concentrating sequence in $H_0^1(B)$, then

$$\lim_{n \rightarrow +\infty} \int_B (e^{4\pi u_n^2} - 1) |x|^\alpha dx \leq \frac{2}{\alpha + 2} \pi e; \quad (10)$$

- 2) give an explicit function $\omega \in H_0^1(B)$ such that

$$\int_B (e^{4\pi \omega^2} - 1) |x|^\alpha dx > \frac{2}{\alpha + 2} \pi e.$$

1) Upper bound. Using the notion of spherical symmetrization with respect to the measure $\mu_\alpha = \int_B |x|^\alpha$ introduced in Section 2, and Theorem 2.1 of Schulz-Vera de Serio, it suffices to show that

$$\overline{\lim}_{n \rightarrow +\infty} \int_{B_\alpha^*} (e^{4\pi |u_{\alpha,n}^*|^2} - 1) dx \leq \frac{2}{\alpha + 2} \pi e$$

where $\{u_{\alpha,n}^*\}$ is the rearranged sequence of u_n , with $\|u_{\alpha,n}^*\| \leq \|u_n\| = 1$, and B_α^* is the ball centered in 0 such that $|B_\alpha^*| = \mu_\alpha(B)$, that is,

$$B_\alpha^* = B \left(0, \sqrt{\frac{2}{\alpha + 2}} \right).$$

Let us set $z_n = \frac{u_{\alpha,n}^*}{\|u_{\alpha,n}^*\|}$; then

$$\int_{B_\alpha^*} \left(e^{4\pi|u_{\alpha,n}^*|^2} - 1 \right) dx \leq \int_{B_\alpha^*} \left(e^{4\pi z_n^2} - 1 \right) dx,$$

so that it suffices to prove that for any radial normalized concentrating sequence in $B(0, \sqrt{\frac{2}{\alpha+2}})$ the upper bound (10) holds. First, we perform a change of variable to reduce the domain to the unit ball. Let $R = \sqrt{\frac{2}{\alpha+2}}\rho$, and $y_n(\rho) = z_n(\sqrt{\frac{2}{\alpha+2}}\rho)$; then

$$2\pi \int_0^{\sqrt{\frac{2}{\alpha+2}}} \left(e^{4\pi z_n^2} - 1 \right) R dR = 2\pi \frac{2}{\alpha+2} \int_0^1 \left(e^{4\pi y_n^2} - 1 \right) \rho d\rho$$

and

$$1 = \int_{B_\alpha^*} |\nabla z_n^*|^2 dx = 2\pi \int_0^1 |y_n'|^2 \rho d\rho.$$

The proof now reads exactly as in [dFdOR] (proof of Theorem 4, step 1), so we can omit it. See also [CC].

2) An explicit function. In this step we exhibit an explicit function $\omega(x)$ such that

$$\int_B \left(e^{4\pi\omega^2} - 1 \right) |x|^\alpha dx > \frac{2}{\alpha+2}\pi e;$$

since, by step 1), any maximizing sequence (if exists) must satisfy $S(\alpha, 4\pi) = \lim_{n \rightarrow +\infty} \int_B (e^{4\pi u_n^2} - 1) |x|^\alpha dx \leq \frac{2}{\alpha+2}\pi e$, we can conclude that $S(\alpha, 4\pi)$ is attained. From now on we assume that u is a generic radial function, and set

$$\varepsilon = \frac{2}{\alpha+2}. \quad (11)$$

As in [SS], following an idea of Smets, Su and Willem, define the new function

$$v(\rho) = \frac{1}{\sqrt{\varepsilon}} u(\rho^\varepsilon); \quad (12)$$

then

$$\int_B |\nabla u|^2 dx = 2\pi \int_0^1 |u'|^2 r dr = 2\pi \int_0^1 |v'|^2 \rho d\rho \quad (13)$$

and,

$$\int_B \left(e^{4\pi u^2} - 1 \right) |x|^\alpha dx = 2\pi\varepsilon \int_0^1 \left(e^{4\pi\varepsilon v^2} - 1 \right) \rho d\rho. \quad (14)$$

We can now perform the change of variable introduced by Moser [M], which transform the radial integral on $[0, 1)$ into an integral on the half-line $[0, +\infty)$,

$$\rho = e^{-t/2} \quad \text{and} \quad w(t) = \sqrt{4\pi}v(\rho);$$

we obtain (recalling the definition (11))

$$\int_B \left(e^{4\pi u^2} - 1 \right) |x|^\alpha dx = \pi \frac{2}{\alpha + 2} \left(\int_0^{+\infty} e^{\frac{2}{\alpha+2} w^2 - t} dt - 1 \right) \quad (15)$$

with

$$\int_B |\nabla u|^2 dx = \int_0^{+\infty} |w'(t)|^2 dt.$$

Following [CC], take $w : [0, +\infty) \rightarrow \mathbb{R}$ to be

$$w(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \leq t \leq 2 \\ \sqrt{t-1} & \text{if } 2 \leq t \leq 1+e^2 \\ e & \text{if } t \geq 1+e^2 \end{cases}.$$

Then, by direct inspection

$$\int_0^{+\infty} |w'(t)|^2 dt = 1$$

and

$$\begin{aligned} \int_0^{+\infty} e^{\frac{2}{\alpha+2} w^2 - t} dt &= \int_0^2 e^{\frac{2}{\alpha+2} \frac{t^2}{4} - t} dt + e^{-\frac{2}{\alpha+2}} \int_2^{1+e^2} e^{-(1-\frac{2}{\alpha+2})t} dt \\ &\quad + e^{e^2 \frac{2}{\alpha+2}} \int_{1+e^2}^{+\infty} e^{-t} dt \\ &= \int_0^2 e^{\frac{2}{\alpha+2} \frac{t^2}{4} - t} dt \\ &\quad + \frac{1}{e} \left[e^{-\frac{\alpha}{\alpha+2} e^2} - \frac{\alpha+2}{\alpha} e^{-\frac{\alpha}{\alpha+2} e^2} + \frac{\alpha+2}{\alpha} e^{-\frac{\alpha}{\alpha+2}} \right] \\ &= \int_0^2 e^{\frac{2}{\alpha+2} \frac{t^2}{4} - t} dt + A_\alpha \end{aligned} \quad (16)$$

where

$$A_\alpha = \frac{1}{e} \left[-\frac{2}{\alpha} e^{-\frac{\alpha}{\alpha+2} e^2} + \frac{\alpha+2}{\alpha} e^{-\frac{\alpha}{\alpha+2}} \right]$$

Let us now estimate the right hand side of (16), when $\alpha \rightarrow 0$. Set $s = -\frac{t}{\alpha+2} + 1$ in the integral term; then

$$\begin{aligned} \int_0^2 e^{\frac{2}{\alpha+2} \frac{t^2}{4} - t} dt &= (\alpha+2) e^{-\frac{\alpha+2}{2}} \int_{\frac{\alpha}{\alpha+2}}^1 e^{\frac{\alpha+2}{2} s^2} ds \\ &> (\alpha+2) e^{-\frac{\alpha+2}{2}} \int_{\frac{\alpha}{\alpha+2}}^1 e^{s^2} \left(1 + \frac{\alpha}{2} s^2 \right) ds \\ &= (\alpha+2) e^{-\frac{\alpha+2}{2}} \left(1 - \frac{\alpha}{4} \right) \int_{\frac{\alpha}{\alpha+2}}^1 e^{s^2} ds + \\ &\quad (\alpha+2) e^{-\frac{\alpha+2}{2}} \frac{\alpha}{4} \left\{ e - \frac{\alpha}{\alpha+2} e^{(\frac{\alpha}{\alpha+2})^2} \right\} \end{aligned}$$

$$\begin{aligned}
&= (\alpha + 2)e^{-\frac{\alpha+2}{2}}\left(1 - \frac{\alpha}{4}\right) \int_0^1 e^{s^2} ds \\
&\quad - (\alpha + 2)e^{-\frac{\alpha+2}{2}}\left(1 - \frac{\alpha}{4}\right) \int_0^{\frac{\alpha}{\alpha+2}} e^{s^2} ds + B_\alpha,
\end{aligned} \tag{17}$$

where

$$B_\alpha = (\alpha + 2)e^{-\frac{\alpha+2}{2}} \frac{\alpha}{4} \left\{ e - \frac{\alpha}{\alpha + 2} e^{(\frac{\alpha}{\alpha+2})^2} \right\}$$

When $\alpha \rightarrow 0$,

$$(\alpha + 2)e^{-\frac{\alpha+2}{2}}\left(1 - \frac{\alpha}{4}\right) = \frac{2}{e} + o(1),$$

$$\int_0^{\frac{\alpha}{\alpha+2}} e^{s^2} ds < \frac{\alpha}{\alpha + 2} e^{(\frac{\alpha}{\alpha+2})^2} = o(1)$$

and

$$B_\alpha = \frac{\alpha}{2} + o(\alpha).$$

Combining (17) with the last estimates yields

$$\int_0^2 e^{\frac{2}{\alpha+2}w^2-t} dt = \frac{2}{e} \int_0^1 e^{s^2} ds + o(1). \tag{18}$$

On the other hand,

$$\begin{aligned}
A_\alpha &= \frac{1}{e} \left[-\frac{2}{\alpha} e^{-\frac{\alpha}{\alpha+2}} e^2 + \frac{\alpha+2}{\alpha} e^{-\frac{\alpha}{\alpha+2}} \right] \\
&= e + o(1) \text{ as } \alpha \rightarrow 0
\end{aligned} \tag{19}$$

Inserting (18) and (19) in (16) we obtain

$$\int_0^{+\infty} e^{\frac{2}{\alpha+2}w^2-t} dt = e + \frac{2}{e} \int_0^1 e^{s^2} ds + o(1);$$

therefore, if ω is the radial function which corresponds to $w(t)$, by (15) we have

$$\begin{aligned}
\int_B \left(e^{4\pi\omega^2} - 1 \right) |x|^\alpha dx &= \pi \frac{2}{\alpha+2} \left(e + \frac{2}{e} \int_0^1 e^{s^2} ds - 1 + o(1) \right) \\
&> \frac{2}{\alpha+2} \pi e \quad \text{when } \alpha \rightarrow 0,
\end{aligned}$$

since $\frac{2}{e} \int_0^1 e^{s^2} ds > 1$, as one can verify estimating the integral with lower Riemann sum, as in [CC] (the value obtained is $\frac{2}{e} \int_0^1 e^{s^2} ds \approx \frac{2.723}{e} > 1$), or expanding the integrand in power series, as in [SS] (here $\frac{2}{e} \int_0^1 e^{s^2} ds \approx \frac{2.906}{e} > 1$). \square

Remark 1. The notion of symmetrization with respect to the measure $\mu_\alpha = \int |x|^\alpha$ is a fundamental tool in the proof of Theorem 1.1, as remarked

in the introduction, since it allows to reduce the variational problem to a one-dimensional problem, as in the unweighted case $\alpha = 0$. Furthermore, it gives a geometric interpretation of the change of variable (12), originally introduced by Smets, Su and Willem in [SSW], which allows to reduce the weighted integral $\int_B (e^{4\pi u^2} - 1) |x|^\alpha dx$ to the unweighted integral $\varepsilon \int_B (e^{4\pi \varepsilon u^2} - 1) dx$ if u is a radial function. Indeed, let us consider a rearranged function $u(r)$; then, by (7) of the previous section (and using the notation (11) for simplicity)

$$u_\alpha^*(r) = u\left(r^\varepsilon \varepsilon^{-\frac{\varepsilon}{2}}\right),$$

so that

$$\begin{aligned} \int_B |\nabla u|^2 dx &= 2\pi \int_0^1 |u'(s)|^2 s ds \\ &= \frac{2\pi}{\varepsilon} \int_0^{\sqrt{\varepsilon}} |u_\alpha^{*'}(r)|^2 r dr \end{aligned}$$

and

$$\begin{aligned} \int_B (e^{4\pi u^2} - 1) |x|^\alpha dx &= 2\pi \int_0^1 (e^{4\pi u^2} - 1) s^{\alpha+1} ds \\ &= 2\pi \int_0^{\sqrt{\varepsilon}} (e^{4\pi |u_\alpha^*|^2} - 1) r dr. \end{aligned}$$

Now, set $r = \sqrt{\varepsilon} \rho$ and $v(\rho) = u_\alpha^*(\sqrt{\varepsilon} \rho)$; then

$$\frac{2\pi}{\varepsilon} \int_0^{\sqrt{\varepsilon}} |u_\alpha^{*'}(r)|^2 r dr = \frac{2\pi}{\varepsilon} \int_0^1 |v'(\rho)|^2 \rho d\rho \quad (20)$$

and

$$2\pi \int_0^{\sqrt{\varepsilon}} (e^{4\pi |u_\alpha^*|^2} - 1) r dr = 2\pi \varepsilon \int_0^1 (e^{4\pi v^2} - 1) \rho d\rho. \quad (21)$$

Equalities (20) and (21) can be restated as

$$\begin{aligned} \int_B |\nabla u|^2 dx &= 2\pi \frac{1}{\varepsilon} \int_0^1 |v'|^2 \rho d\rho, \\ \int_B (e^{4\pi u^2} - 1) |x|^\alpha dx &= 2\pi \varepsilon \int_0^1 (e^{4\pi v^2} - 1) \rho d\rho, \end{aligned}$$

where

$$v(\rho) = u_\alpha^*(\sqrt{\varepsilon} \rho) = u(\rho^\varepsilon);$$

this is exactly the change of variable introduced by Smets, Su and Willem in [SSW], and differs from (12) by a dilation factor. Therefore, the change of variable $\rho = r^\varepsilon$ performed to obtain asymptotic estimates for the radial supremum $S^{\text{rad}}(\alpha, 4\pi)$ in [SS] (respectively $S^{\text{rad}}(\alpha, p)$ in [SSW]) coincides with

the spherical symmetrization with respect to the measure μ_α (rescaled so to reduce the symmetrized domain B_α^* to the unit ball B).

Remark 2. Note that we have proved step 2 testing with a radial function. It easy to show that if $\alpha \rightarrow +\infty$, a function $w(x)$ such that

$$\int_B (e^{4\pi w^2} - 1) |x|^\alpha dx > \frac{2}{\alpha + 2} \pi e,$$

if exists, must be *non radial*. Indeed, for any u radial function in $H_0^{1,\text{rad}}(B)$, $\int_B (e^{4\pi u^2} - 1) |x|^\alpha dx = 2\pi \varepsilon \int_0^1 (e^{4\pi \varepsilon v^2} - 1) r dr$ by (14); but

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} \left(2\pi \int_0^1 (e^{4\pi \varepsilon v^2} - 1) r dr \right) &= \frac{1}{\varepsilon} 2\pi \int_0^1 4\pi \varepsilon v^2 e^{4\pi \varepsilon v^2} r dr \\ &> \frac{1}{\varepsilon} 2\pi \int_0^1 (e^{4\pi \varepsilon v^2} - 1) r dr, \end{aligned} \quad (22)$$

since $te^t > e^t - 1$ for every $t > 0$ (note that the inequality is strict, and there is equality if and only if $t = 0$). Integrating the previous inequality yields

$$\begin{aligned} 2\pi \int_0^1 (e^{4\pi v^2} - 1) r dr &> \frac{1}{\varepsilon} 2\pi \int_0^1 (e^{4\pi \varepsilon v^2} - 1) r dr \\ &= \frac{1}{\varepsilon^2} \int_B (e^{4\pi u^2} - 1) |x|^\alpha dx. \end{aligned}$$

Therefore, if there exists a radial function $w(x)$, with $\|w\| \leq 1$, such that $\int_B (e^{4\pi w^2} - 1) |x|^\alpha dx > \frac{2}{\alpha+2} \pi e = \varepsilon \pi e$, we have also

$$S(0, 4\pi) \geq \int_B (e^{4\pi w^2} - 1) dx > \frac{1}{\varepsilon^2} \int_B (e^{4\pi w^2} - 1) |x|^\alpha dx > \frac{1}{\varepsilon} \pi e;$$

this implies that $S(0, 4\pi)$ is unbounded as $\alpha \rightarrow +\infty$, that is a contradiction. Hence, the problem of the existence of a maximizer for $S(\alpha, 4\pi)$, when $\alpha > \alpha_*$, is reduced to finding a non-radial function w satisfying the lower bound $\int_B (e^{4\pi w^2} - 1) |x|^\alpha dx > \frac{2}{\alpha+2} \pi e$ (trying to adapt the proof presented here).

Remark 3. It remains an open problem whether the supremum $S(\alpha, 4\pi)$ is attained for every $\alpha > 0$.

References

- [Ba] B. Bandle, *Isoperimetric inequalities and applications*, Pitman monographs and surveys in pure and applied mathematics **0007**, Pitman Advanced Publishing Program, Boston, 1980.
- [CC] L. Carleson and A. Chang, *On the existence of an extremal function for an inequality of J. Moser* Bull. Sci. Math. (2) **110** (1986), 113-127.

- [CT] M. Calanchi and E. Terraneo, *Non-radial maximizers for functionals with exponential nonlinearity in $(R)^2$* Adv. Nonlinear Stud. **5** (2005), 337-350.
- [dFdOR] D. G. De Figueiredo, J. M. do Ó and B. Ruf, *On an inequality by N. Trudinger and J. Moser and related elliptic equations* Comm. Pure Appl. Math. **55** (2002), 135-152.
- [F] M. Flucher, *Extremal functions for the Trudinger-Moser inequality in 2 dimensions* Comm. Math. Helv. **67** (1992), 471-479.
- [HK] P. Hess and T. Kato, *On some linear and nonlinear eigenvalue problems with an indefinite weight function* Comm. Partial Differential Equations **5** (1980), 999-1030.
- [K] B. Kawohl, *Rearrangements and convexity of level sets in PDE*, Lecture notes in Mathematics **1150**, Springer Verlag, Berlin, 1985.
- [L] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case, part 1* Riv. Mat. Iberoamericana **1** (1985), 145-201.
- [M] J. Moser, *A sharp form of an inequality by N. Trudinger* Indiana Univ. Math. J. **20** (1970/71), 1077-1092.
- [Po] S. I. Pohozaev, *The Sobolev embedding in the case $pl = n$* , Proceedings of the Technical Scientific Conference on Advances of Scientific Research 1964-1965. Mathematics Section, p. 158-170, Moskov. Energet. Inst., Moscow, (1965).
- [SS] S. Secchi and E. Serra, *Symmetry breaking results for problems with exponential growth in the unit disk*, to appear on Commun. Contemp. Math.
- [SSW] D. Smets, J. Su and M. Willem, *Non-radial ground states for the Hénnon equation*, Commun. Contemp. Math. **4** (2002), 467-480.
- [SVS] F. Schulz and V. Vera de Serio, *Symmetrization with respect to a measure*, Trans. Amer. Math. Soc. **337** (1993), 195-210.
- [SW] D. Smets and M. Willem, *Partial symmetry and asymptotic behavior for some elliptic variational problems*, Calc. Var. Partial Differential Equations **18** (2003), 57-75.
- [Ta] G. Talenti, *A weighted version of a rearrangement inequality*, Ann. Univ. Ferrara Sez. VII (N.S.) **43** (1997), 121-133.
- [Tr] N. S. Trudinger, *On imbedding into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473-484.